The Primordial Density Perturbation
Errata

Updated 23rd May 2011
Previous update 6th February 2011
Changes since last update marked with ***

The following are errata that we know about. In some places we also give additional sentences, where the text as it stands would be ambiguous or wrong. Please let us know if you find any other errors or ambiguities.

Chapter 2

The integral on the right hand side of Eq. (2.32) goes over the boundary of $\mathcal{V}$, usually denoted by $\partial\mathcal{V}$.

Below Eq. (2.37), *replace* point in spacetime point *by* point in spacetime

In the first line of page 17, *replace* frame frame *by* frame

Complete the last sentence of the paragraph containing Eq. (2.57) by:
... within a cell d^3p centered on $\mathbf{p}$.

In line 3 of Section 2.6.2, *replace* a *by* `a`

At the end of the first sentence of the second paragraph of Section 2.6.2 *insert*
,from particles whose momentum is within an infinitesimal region d^3p centered on momentum $\mathbf{p}$

Chapter 3

In the line after Eq. 3.10, *replace* $\partial x^\mu/\partial x'^\nu$ *by* $\partial x'^\nu/\partial x^\mu$

In Section 3.4, paragraph 1, line 7 *replace* diagonal *by* diagonal with elements ±1

In Eqs. (3.41) and (3.42), multiply the right hand side by -1

Three lines below Eq. (3.47), *replace* *** enought *by* enough

On page 39 replace the content of footnote 7 by:
The homogeneity and isotropy is verified by observation within a region around us (the observable Universe) and is usually supposed to hold in some bigger region which we call simply the Universe. We will use 'universe' without the capitalization to denote things different from the Universe, notably the separate universes of Section 5.3.2.

On page 39, paragraph 2, *replace* observable Universe *by* Universe

Two lines after Eq. 3.81, *replace* orthogonal *by* orthonormal

Three lines after Eq. (3.85), *replace* twice *by* be half

Chapter 4

In Eq. (4.4), *replace* f *by* f_a

In the line before Eq. (4.10), *replace* $\rho = 3P$ *by* because we are considering $\rho = \rho_\mathrm{r}$, which

On page 54, line 14, *replace* a *by* $1/a$

At the end of Eq. (4.14) *replace* .. *by* .

In Eq. (4.28), *replace* (- *by* -(

In Section 4.4.3, paragraph 2, lines 4 and 7, *replace* He^4 *by* $^4\mathrm{He}$

On page 62, eleven lines from the end, *replace* Λ *by* ρ_vac

On the line after Eq. (4.61), *replace* 11 *by* 9

Chapter 5

Before Eq. (5.3) *replace*
average
by
spatial average energy density, which we denote by $\rho_\mathrm{a}(t)$

Delete the sentence following Eq. (5.3)

On page 73, five lines before the end of Section 5.1.3, *replace* $\delta\rho$ *by* $\delta\rho/\rho$

On line 1 of page 77, *delete* observable

In Eq. (5.7), *replace* e^h *by* e^{2h}

In Eq. (5.8), divide the left hand side by $\mathcal{V}$

Delete the last two sentences of Section 5.4.1

In Eq. (5.13), *replace* $\tilde{h}$ *by* $2\tilde{h}$

Four lines after Eq. (5.13), *replace* comoving slicing *by* comoving threading

Replace the sentence containing Eq. (5.14) by:
The perturbation in the local number of Hubble times between two generic slices is

$$\delta N_{12}(\mathbf{x}) = \delta\int_{t_1}^{t_2}\frac{1}{a}\frac{da}{dt}dt = \psi(\mathbf{x},t_2) - \psi(\mathbf{x},t_1). \qquad (1)$$

In Eqs. (5.17) to (5.19), *replace* $\widetilde{\delta f}(\mathbf{x},t)$ *by* $\widetilde{\delta f}(\mathbf{x},\tilde{t})$

In the second line of Section (5.5.2), *replace*
From Eq. (5.24),
by
To first order, $\tilde{t}(\mathbf{x}, t-\delta t(\mathbf{x},t)) = t$. Therefore, the time shift going from a uniform density slice at time t to a generic slice at time $\tilde{t} = t$ is $-\delta t$. Using Eq. (5.14) with $t_2 - t_1 = \delta t$,

After Eq. (5.23) *insert*
(From now on we are dropping the tilde on the time coordinate of the generic slicing.)

Delete the sentence after Eq. 5.26

Replace the sentence before Eq. (5.27) by:
Suppose also that each component has a unique relation $P_{\rm a}(\rho_{\rm a})$. Then there are separately conserved quantities

*** In the line after Eq. (5.28), *replace* $i = \gamma$ *by* $\mathtt{a} = \gamma$

*** In Exercise 5.3, *replace* Eq. (5.10) *by* Eq. (5.26)

Chapter 6

In the line before Eq. (6.3), *replace*
The choice of labeling doesn't matter.
by
If we allow the index too have either sign and choose $\mathbf{k}_{-n} = -\mathbf{k}_n$, the reality of $g(\mathbf{x})$ is equivalent to $g_n^* = g_{-n}$.

After Eq. (6.3) *insert*
If the $k = 0$ mode (spatial average) is nonzero, it is ignored in the context of stochastic properties. It can be absorbed into the background unless it corresponds to anisotropy.

In the line before Eq. (6.12) *delete*
determined by the reality condition, to be

In Eq. (6.22), the integral over d^3k should not specify the range 0 to ∞.

In the two lines after Eq. (6.22), *replace* σ^2 *by* σ_g^2

In Eqs (6.32) and (6.33), *replace* $(2\pi)^3$ *by* $(2\pi)^6$

In Eq. (6.38), *replace* $(2\pi)^{-6}$ *by* $(2\pi)^{-9}$

Multiply the right hand side of Eq. (6.42) by $(2\pi)^{-3}$

In the footnote on page 95, *replace* P_{gn} *by* P_{g_n}

Three lines before Eq. (6.64), *replace* saussian *by* gaussian

In Eq. (6.64), *replace* $\sqrt{2\pi}\sigma_n$ *by* $2\pi\sigma_n^2$

In Eq. (6.71), *replace* $\mathbf{k}_i$ *by* k_i

After Eq. (6.71), *delete* where g is gaussian

In Eqs. (6.74) and (6.75), delete the factor 3/5

Two lines after Eq. (6.75), *replace* $3f_{\rm NL}/5$ *by* $5f_{\rm NL}/3$

In Exercise 6.4, *replace* $3f_{\rm NL}/5$ *by* $5f_{\rm NL}/3$

Chapter 8

Page 118. In writing Eq. (8.7) we are taking the polarization tensors to be fixed, as is explained in footnote 2 of Chapter 12

Replace the first 6 lines of page 123 by:
We also expect the right hand sides of Eqs. (8.35) and (8.36) to be negligible. Then, taking k/aH to be negligible in Eqs.(8.33) and (8.44) (or more simply, in the 0-0 component of the Einstein equation) we find $\delta = -2(\Psi + (aH)^{-1}\dot{\Phi}$ leading to

In Figure 8.2 *replace* Mpc ^{1} *by* Mpc^{-1}

In Eq. (8.57), *replace* $\dot{R}$ *by* R

In Exercise 8.3, *replace* dominated *by* domination

Chapter 9

In Figure 9.2, *replace* Mpc ^{1} *by* Mpc^{-1}

In Figure 9.2, *replace* 10^3 *by* 10^{-3}

In Figure 9.2, *replace* 10^4 *by* 10^{-4}

In the paragraph below Eq. (9.30) *replace* Eq. (9.28) *by* Eq. (9.29)

In Eq. (9.32), *replace* m *by* M

Three lines after Eq. (9.35), *replace* 'the the' *by* 'the'

Chapter 10

On page 157 replace Eq. (10.17) and the preceding line by:
corresponds to Fourier modes with wavelength

$$\Delta\theta = 2\pi/\ell.$$

This is the angular scale in the sky, explored by a given ℓ.

In Eq. (10.42), *replace* 1/5 *by* $-1/5$

Three lines below Eq. (10.46), *replace* $k < k_{\rm vl}$ *by* $k > k_{\rm vl}$

In Eq. (10.56), delete the factor 4π

In Example 10.2, *replace* θ *by* Θ

Chapter 11

In the line before Eq. (11.8), *replace* $\mathbf{p}/a(\eta)$ *by* $\mathbf{p}a(\eta)$

In Eq. (11.39), delete the factor 2

On the right hand side of Eq. (11.45), *replace* U' *by* U

In the last line of page 180, delete the first appearance of ± 2

In the last line of page 180 *delete* the notation

In the line after Eq. (11.55), *replace* $C[\Theta(\mathbf{n}]$ *by* $C[\Theta(\mathbf{n})]$

Four lines before Eq. (11.65), *replace* θ *by* Θ

Chapter 12

In line 1 of page 191, *replace* $=$b *by* $= B$.

In Eq. (12.15), *replace* 3 *by* 6

In last line of page 201, *replace* Eqs. (11.61) and (11.61) *by* Eqs. (11.60) and (11.61)

In Eq. (12.33), *replace* 4 *by* 16

Divide the right hand side of Eq. (12.37) by 4

Chapter 13

At the end of Section 13.2.2 insert this paragraph:
In writing the Hamiltonian equations (13.15)-(13.17), we assume that the equations $p_n = \partial L/\partial \dot{q}_n$ can be solved to give $\dot{q}_n(q_1, p_1, q_2, \ldots, t)$ as functions of independent variables p_n. If that is not the case, there are constraint equations relating the p_n. The system of harmonic oscillators is obviously unconstrained, and since this will be our only detailed application of the Hamiltonian formalism we will ignore the issue of constraints.

*** Multiply the first term of Eq. (13.49) by -1.

Chapter 14

Labeling of scalar fields: in this chapter they are denoted variously as ϕ_n, ϕ_i and ϕ_a. A consistent notation ϕ_n throughout would be preferable.

*** *replace the two sentences after Eq. (14.9) by*
Without the last term this has the two symmetries, $U_A(1)$ with $q_1 = 1$ and $q_2 = 0$ and $U_B(1)$ with $q_1 = 0$ and $q_2 = 1$, and the full symmetry is $U_A(1) \otimes U_B(1)$. Including the last term it has the single symmetry $U_C(1)$ with $q_1 = 1$ and $q_2 = -2$.

*** In line 7 of the paragraph after Eq. (14.18), *replace* $N \times N$ matrices $\hat{T}_n$ *by* three $N \times N$ matrices $\hat{T}_a$

In the line before Eq. (14.23), *delete* consider

In line 9 of page 235, *replace* components *by* component

In the line after Eq. (14.29), *replace* $\sqrt{2}$ *by* $\sqrt{2}\lambda$

After Eq. (14.31), *replace* $\langle\phi^2\rangle$ *by* $\langle\phi\rangle^2$

In the fourth line before the end of page 239, *delete* at

In the second line before the end of page 239, *replace* λ_a *by* λ

In the second line before the end of page 239, *replace* ϕ *by* ϕ_n

In Eq. (14.39), *replace* $+V$ *by* $-V$

In the penultimate paragraphs of Section 14.7, *replace* spacetime *by* spacetime position

In the second and third lines of Section 14.8, *delete*
As with the global case, the complex field (or fields) whose vev spontaneously breaks a gauge symmetry.

The notation $F^2_{\mu\nu}$, introduced in Eq. (14.53), means $F_{\mu\nu}F^{\mu\nu}$

*** The notation $(\partial_\mu\chi)^2$, introduced in Eq.(14.54) means $(\partial^\mu\chi)(\partial_\mu\chi)$

The notation $W^2_{\mu\nu}$, introduced in Eq. (14.55), means $W_{\mu\nu}W^{\mu\nu}$

The notation $(W^i_\mu)^2$, introduced in Eq (14.56), means $W^i_\mu W^{i\mu}$

In Exercise 14.7, *replace* supersymmetry breaking *by* spontaneous symmetry breaking

Chapter 15

*** In the line before Eq. (15.9) *replace* (15.7) *by* (15.8)

Replace
Eq. (15.9)
by
$\frac{\partial \hat{H}}{\partial \hat{q}_n} = -i[\hat{H}, \hat{p}_n], \qquad \frac{\partial \hat{H}}{\partial \hat{p}_n} = i[\hat{H}, \hat{q}_n].$

*** Complete the paragraph containing Eq. (15.9) in the following way:
... appear has in general to be specified. For this reason one has in general to be careful about the definition of the derivatives $\partial/\partial\hat{q}_n$ and $\partial/\partial\hat{p}_n$ acting on an operator. We will not pursue this question, since we perform calculations only for the system of harmonic oscillators for which there is no ambiguity.

On page 251, from the heading **Discrete symmetry** to the end of the Section, *replace* $\hat{U}$ *by* $\hat{V}$

*** The expression (15.22) for $\hat{H}$ should be multiplied by ω

After Eq. (15.39) *replace*
Since the vacuum ..... pressure is $-\langle V\rangle$
by
Using Eq. (13.32), a similar calculation for the pressure gives $\langle P\rangle = \frac{1}{3}\langle\rho\rangle$. These vacuum contributions to ρ and P have to be ignored to get a viable cosmology. Note that they cannot be absorbed into the cosmological constant, which has $P = -\rho$. The discrepancy between the vacuum contributions and the cosmological constant arises because the cutoff $\Lambda_{\rm UV}$ on 3-momentum breaks Lorentz invariance,

In the third line from the end of Section 15.4.3, *replace* through *by* 'defined through'

On page 260, eleven lines from the bottom, *replace* e^{iq_ψ} *by* $e^{iq_\psi\lambda}$

On page 261, one line before the end of Section 15.6, *replace* e^{-iq_R} *by* $e^{-iq_R\lambda}$

On page 261, one line before the end of Section 15.6, *replace* e^{iq_R} *by* $e^{iq_R\lambda}$.

On page 261, on the last line of Section 15.6, *replace* e^{iq}

by $e^{iq\lambda}$

*** On the line before Eq. (15.62), *replace* Eq. (15.36) *by* Eq.(15.29)

In the second and third lines after Eq. (15.62) *replace* the late-time occupation number *by* the expectation value of the late-time occupation number

In line 2 of page 265, *replace* some the *by* some of the

** In line 2 of page 266, delete the factor 2.

After Eq. (15.68), *replace* $\phi \gg m_\chi$ *by* $\sqrt{g}\phi \gg m_\chi$

After Eq. (15.69), *replace* (13.38) *by* (15.68)

*** In the footnote on page 266, *replace* g^2 *by* g

*** In exercise 15.1 *replace* classical expression (13.7) *by* (15.8) (using Eq. (15.7)).

*** In exercise 15.2, *replace* $[\hat{a}^\dagger, \hat{a}] = 1$ *by* $[\hat{a}, \hat{a}^\dagger] = 1$

*** In exercise 15.5 *replace* the late-time occupation number *by* the expectation value of the late-time occupation number

Chapter 16

On page 270, the paragraph before Eq. (16.1) should continue as follows:
so are the similarly-defined products QH and QH_c.

At the end of the paragraph containing Eq. (16.1), *insert* Extending the notation of Eq. (15.57), the gauge invariant kinetic term for L is $L^* \not{D} L \equiv L_i^* \not{D} L_i$, and similarly for Q.

In Eqs. (16.4) and (16.23), the sign in front of the term containing y^u should be minus instead of plus

In Eq. (16.27), *replace* 10^{-2} *by* 10^{-6}

In the middle of page 283, *replace* 100 eV *by* 10^{-2} eV

Chapter 17

In the line after Eq. (17.5), *replace* as in Eq. (15.53) *by* as in Section (15.6.1)

Two lines after Eq. (17.11), *replace* $e^{i\theta\phi}$ *by* $e^{i\theta}\phi$

Eq. (17.12), and the sentence containing it, should be at the end of the paragraph not at the beginning.

In the line after Eq. (17.15), *replace* $m_{\chi_1}^2 + m_{\chi_1}^2$ *by* $m_{\chi_1}^2 + m_{\chi_2}^2$

In Eq. (17.26), the final M_S^4 should be M_S^2.

Chapter 18

*** In line 5 page 310, complete the sentence in the following way:
... more than enough to remove any conflict with observation.

*** In line 17 page 311 *replace* anisotropy *by* isotropy

** In line 18 page 311 *replace* homogeneity *by* inhomogeneity

On page 312, line 6, *replace* known to exist only *by* only known to exist

*** In Eq. (18.7) *replace* ent *by* entry

*** In the line after Eq. (18.15) *replace* (13.56) *by* (13.57)

*** Two lines after Eq. (18.25) *replace* $\ddot{a} > 1$ *by* $\ddot{a} > 0$

*** In the second line of Section 18.6.2, *replace*
(solutions of Eq. (13.57))
by
(solutions of Eq. (13.57) with H given by the Friedman equation)

*** In Fig. 18.4 caption, *replace* (18.35) *by* (18.34)

*** Three lines after Eq. (18.39) *replace* m_χ^2 *by* m_χ

In the line before Eq. (18.40), *replace* Eq. (18.35) *by* Eq. (18.34)

*** In Ex. 18.5 *replace*
Use your solution for ϕ to calculate the time at which inflation ends
by
Use your solution for ϕ to calculate $H(t)$ and hence $a(t)$

*** In Ex. (18.6) *replace* (18.35) *by* (18.34)

Chapter 19

Page 326: in the last line of the first paragraph, *replace* is called the *by* as

Chapter 20

*** Three lines before Eq. (20.1) *replace* V_n *by* $V_n \equiv \partial V/\partial\phi_n$.

*** After Eq. (20.7) *insert*
(We defined $V_{nm} \equiv \partial^2 V/\partial\phi_n\partial\phi_m$.)

*** [No change the last 2 sentences of page 334 now — see Supplementary Material]

Chapter 21

On page 345, line 9, *replace* the MSSM *by* its extensions

On page 375, line 7, *replace* applies *by* still applies

Chapter 23

On page 378, footnote 5, *replace* we absorb the vacuum fluctuation into the potential *by* we ignore the vacuum fluctuation

Chapter 24

*** In Eq. (24.6) *delete* $(2\pi)^3$

*** In Eq. (24.6) *replace* t *by* η

*** On the left hand side of Eq. (24.10) *insert* $(2\pi)^3$

*** Before Eq. (24.15) *replace* ϕ *by* $\delta\phi$

*** In Eq. (24.18) and the following line *replace* $\hat{H}$ *by* H

At the beginning of paragraph 2, Section *** 24.3.2, *replace*
except where stated
by
defined by $D = 0$ where D is defined by *** Eq. (8.8). (This generalizes to all scales the super-horizon definition of 'flat slicing' given after Eq. (5.14).)

Divide the right hand sides of Eqs. (24.30) and (24.31) by $M_{\rm Pl}^2$

Multiply the right hand side of Eq. (24.43) by $-1/3$.

Delete the sentence after Eq. (24.47)

Insert a full stop at the end of the paragraph containing Eq. (24.47)

In line 3 of Section 24.5.1, *replace* Section 24.3.1 *by* Section 24.1.3

Eq. (24.53) and the sentence after it should appear immediately after Eq. (24.50)

In line 3 of Section 24.7, *replace* horizon entry *by* horizon exit

In Eq. (24.29), delete the factor 1/2.

*** In Ex. 24.3 *replace* $\phi_{\rm max} > H$ *by* $\phi_{\rm max} < H$

Chapter 25

In Eq. (25.9), *replace* $M_{\rm Pl}^{-1}$ *by* $M_{\rm Pl}^{-2}$

Multiply the right hand side of Eq. (25.6) by -1

In the line above Eq. (25.10), *replace* (24.61) *by* (24.60)

After the sentence containing Eq. (25.17) *insert*
To make the spatial average of $\zeta(\mathbf{x})$ vanish, the second term should actually be $\frac{1}{2}N''[(\delta\phi(\mathbf{x}))^2 - \overline{(\delta\phi)^2}]$ (where the overline denotes the spatial average) and similarly for higher powers.

Replace Eq. (25.20) *by* $N'' = (2\epsilon - \eta)N'^2$

Replace Eq. (25.21) *by* $\zeta = N'\delta\phi + \frac{1}{2}(2\epsilon - \eta)(N'\delta\phi)^2$

Multiply by -1 the right hand sides of Eqs. (25.22), (25.32) and (25.33), and the equation in the third line of Section 25.4.3

Replace Eq. (25.24) *by* $\frac{6}{5}f_{\rm NL} = 3\epsilon - \eta + \zeta\Delta N + \epsilon f$

In Eq. (25.25), *replace* f *by* $(f + \frac{3}{2})$

After Eq. (25.25) *insert*
It can be shown that $0 \leq f \leq 5/6$, the lower limit corresponding to the squeezed configuration $k_1 \ll k_2 \simeq k_3$ and the upper limit to the equilateral configuration $k_1 = k_2 = k_3$.

Replace Eq. (25.26) *by* $\frac{3}{5}f_{\rm NL}(k_1, k_2, k_3) = -\frac{n-1}{4} + \frac{r}{32}f(k_1, k_2, k_3)$

In the third line of Section 25.4.3, and in Eq.25.33), *replace* $(n-1)/4$ *by* $-(n-1)/4$

Replace the first sentence of page 413 *by* A given location has displacement $a\mathbf{x} = a_L\mathbf{x}_L$, where $\mathbf{x} = [1 + \zeta_+(0)]\mathbf{x}_L$.

On the right hand sides of Eqs. (25.30) and (25.31), *replace* $-$ *by* $+$

Multiply the right hand side of Eq. (25.33) by -1

In line 2 of Section 25.4.4, *replace* horizon entry *by* horizon exit

Two lines before Eq. (25.37) *replace* L *by* L^{-1}

On page 415, line 1, *replace* bracket *by* large bracket

Four lines after Eq. (25.55), *replace* proportional to $1/c_s$ *by* of order $1/c_s^2$

In the penultimate line of page 419 *replace* bispectrum *by* calculated bispectrum for k-inflation

In the third line after Eq. (25.58) *replace* Schr̈odinger *by* Schrödinger

Chapter 26

*** After Eq. (26.3) *insert*
(We defined $N_i \equiv \partial N/\partial\phi_i$.)

*** In Eq. (26.5) *replace* $\sum_i$ *by* $\sum_{ij}$

*** After Eq. (26.5) *replace*
these are weighted means of η_{ii}^2. The weighting is different from the one in Eq. (20.9)
by

$\bar{\bar{\eta}}$ is a weighted mean of η_{ii}. The weighting is different from the one in Eq. (20.8)

*** After Eq (26.8) *insert*
(In these expressions a subscript denotes differentiation with respect to a field, for example $N_{ij} \equiv \partial^2 N/\partial_i \partial_j$, and we invoked the slow-roll approximation $3H\dot{\phi}_i \simeq -V_i$.) To evaluate Eq. (26.6) we can take $\mathcal{P}_\zeta(k)$ from observation. Then, in contrast with .....
(no change in the rest of the sentence)

*** After Eq. (26.9) *delete* Specializing now to Einstein gravity

*** Seven lines after Eq. (26.9) *delete* Specializing further to the case of light fields

*** Section 26.3, second paragraph: after the first sentence *insert*
We will assume that the perturbed fields have canonical kinetic terms.

Multiply the second term of Eq. (26.18) by 1/2.

Divide the right hand side of Eqs. (26.21)–(26.23) by 8.

Four lines after Eq. (26.33) *replace* than *by* more than

In Eqs. (26.26) and (26.34), *replace* m *by* m_σ

Two lines before Eq. (26.35), *delete*)

Chapter 27

On the left hand side of Eq. (27.2) *replace* $t_{\rm cr}$ *by* t

In the line after Eq. (27.4), *replace* $=\equiv$ *by* $=$.

Chapter 28

In the bottom line of page 447, *replace* Rfsg. *by* Refs.

On page 455, the letter V appearing above the top left Figure should appear as the label of the vertical axis of the top right Figure. Also, the digit 0 appearing below the bottom left Figure should be below the bottom right Figure at the position of the minimum of the curve.

Two lines below Eq. (28.12), *replace* of decrease *by* or decrease

At the end of Eq. (28.22) *replace* $M_{\rm Pl}/\phi$ *by* $16\pi^2/g^2$

Two lines below Eq. (28.23), *replace* $M_{\rm GUT}^{1/4}$ *by* $M_{\rm GUT}$

On page 460, line 4, *replace* the required values *by* observation for n

In Eq. (28.26), *replace* M *by* $M_{\rm GUT}$

Before Eq. (28.27), *insert*
to leading order in $M_{\rm GUT}/\phi$

On page 469 the caption of the Figure should say:
... (ii) $\ln V$ concave-down and V concave-up, ...

Appendix A

In Eq. (A.3), *replace* $(2\ell+1)$ *by* $2/(2\ell+1)$

In Eq.. (A.5), replace the square bracket by a square root

In Eq. (A.6), *replace* $(1-x^2)^m$ *by* $(1-x^2)^{m/2}$

In Eq. (A.6), *replace* $2!$ *by* 2^ℓ